Exam Quantum Field Theory
June 22, 2017
Start: 9:00h End: 12:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on one side, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also lose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

The energy projectors for spin 1/2 Dirac fermions:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \frac{\not{p}+m}{2m}$$

$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \frac{\not{p}-m}{2m}$$

$$\{\gamma_5, \gamma_\mu\} = 0 \qquad \gamma_5^2 = \mathbb{1} \qquad \gamma_5^\dagger = \gamma_5 \qquad \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4(g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}) \qquad (d=4)$$

$$\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = -4i\epsilon^{\mu\nu\alpha\beta} \quad \text{(antisymmetric!)} \qquad (d=4)$$

1. [2 points] Given the lagrangian density for two interacting real scalar fields $\phi_{1,2}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_2^2\phi_2^2 - \lambda\phi_1^2\phi_2^2$$

derive the Feynman rule for the interaction vertex and draw the corresponding Feynman diagram in momentum space (solid line for ϕ_1, dash line for ϕ_2). **Show your work**

2. (2 points total) Given the Lagrangian density for the free massive Dirac theory

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\overrightarrow{\partial}_\mu - m\right)\psi$$

a) **[1 points]** Derive the equation of motion (EoM) for the fields ψ and $\bar{\psi}$. **Show your work**

b) **[1 points]** Show that the current $J^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, i.e. $\partial_\mu J^\mu = 0$, by using the EoM for ψ and $\bar{\psi}$. **Show your work**

3. (2 points total) Consider the Dirac lagrangian of problem 2 with an additional four-fermion interaction

$$\mathcal{L}_I = \frac{G}{\sqrt{2}}J_L^\mu J_{L\mu}^\dagger$$

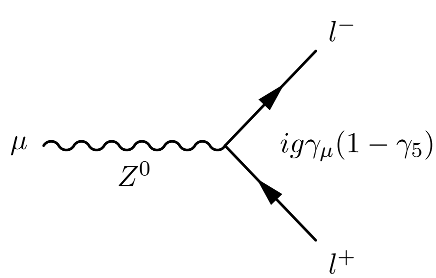
with the left-handed current given by

$$J_L^\mu = \bar{\psi}\gamma^\mu\left(1-\gamma_5\right)\psi\,.$$

a) **[1.5 points]** Derive the formula for the superficial degree of divergence D for this theory in d spacetime dimensions. **Show your work**

b) **[0.5 points]** For which spacetime dimension is this theory renormalizable?

4. (3 points total) The neutral gauge boson Z^0 that mediates weak interactions can decay into a pair of charged leptons $Z^0 \to l^+l^-$. Let us consider a simplified version of the Standard Model interaction vertex given by the Feynman rule



where the external Z^0 (a spin 1 massive particle) carries a real polarization vector $\epsilon_a^\mu(\vec{k})$, with $a = 1, \ldots 3$ and tri-momentum $\vec{k}$.

a) **[2 points]** Calculate the unpolarized squared amplitude $X = \left(\mathcal{A}^\dagger \mathcal{A}\right)_{\text{unpol.}}$ for this decay process at leading order in the coupling g. **Show your work**

b) **[1 points]** Using relativistic kinematics find the decay rate Γ in the centre-of-mass (CM) frame where the decaying particle is at rest, i.e. $s = k^2 = M_Z^2$. **Show your work**

Hints:

- Use for the sum over the Z^0 polarizations

$$\sum_{a=1}^{3} \epsilon_a^\mu(k)\epsilon_a^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M_Z^2}$$

- Use the CM formula for the two-body differential decay rate

$$d\Gamma = \frac{1}{16\pi^2} \frac{m_l^2}{M_Z} \sqrt{1 - \frac{4m_l^2}{M_Z^2}}\, X\, d\Omega$$

  with $d\Omega$ the differential solid angle and X defined in a).

- You should find that X (CM frame) can be written in terms of the lepton mass m_l and M_Z only.